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AN APPLICATION OF SEMI-MARKOV CHAINS TO ASW TACTICAL SYSTEMS

By D.D. Culbertson

CNA Research Contribution No. 118

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AN APPLICATION OF SEMI-MARKOV CHAINS TO ASW TACTICAL SYSTEMS

By D.D. Culbertson

September 26, 1969

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ABSTRACT

An application of semi-Markov chains to Anti Submarine Warfare (ASW) tactical systems is illustrated with a hypothetical example. The example involves estimating the probabilities of prosecuting false contacts for varying lengths of time. The use of Markov chains in the analysis of ASW systems, using Fleet ASW Data Analysis Program (FADAP) data, is discussed briefly.

INTRODUCTION

In measuring the performance of an ASW system, one of the important variables is the degradation in performance caused by the prosecution of false contacts. Also, if the system is degraded, is it because there is a high frequency of false contacts, or is it because the false contacts are prosecuted for excessive lengths of time? The whole sequence of events from detection to attack by ASW units is often analyzed by describing the sequence with Markov chains as in reference (a). It is the purpose of this paper, first, to point out that the use of certain Markov chains can be extended in a natural way to include waiting times in the various states. The waiting time in a state described as "prosecution of a false contact" is interpreted as the time spent prosecuting the false contact. Second, the structure of the FADAP data bank is such that it describes the ASW events sequentially and, therefore, lends itself to analysis of questions such as those posed above by means of Markov chains.

DEFINITIONS AND NOTATIONS

Suppose M denotes a Markov chain with a sequence of random variables M_k. Associated with M is a matrix, P of transition probabilities. The transition probabilities are denoted by

$$p_{ir} = Pr\left\{M_{k+1} = r \mid M_k = i\right\}$$

where p_{ir} is the probability the process will be in state r on k+1 step given that the process is in state i on step k. Now suppose that $t_0 = 0$ and the time spent in state M_k is defined by $t_{k+1} - t_k$. A sequence $\{x_t\}$ where $x_t = M_k$ provided that $t_k \leq t < t_{k+1}$ is called a semi-Markov chain. References (b) and (c) define the substate chain $\{(x_t, y_t)\}$ where $y_t = 0$ if $t = t_k$ and $y_t = t_{k+1} - t$ if $t_k < t < t_{k+1}$ and show that this substate chain is also a Markov chain. A typical sequence in the substate chain might appear as follows:

(i,0), (i,5), (i,4), (i,3), (i,2), (i,1), (j,0), (j,2), (j,1),
(k,0), (m,0), (m,3),...

Let $w_{ij} = Pr\left\{t_{k+1} - t_k = j \mid M_k = i\right\}$. This is the distribution of durations of system state i. Only integer durations are considered, a satisfactory approximation if an appropriate time scale is chosen. The following notation is adopted.

$$b_{ij} = \sum_{n=j+1}^{\infty} w_{in}, \qquad \text{(the probability the duration in state i is greater than j)}$$

$$e_i = \sum_{j=0}^{\infty} jw_{ij}, \qquad \text{(the average duration in state i).}$$

The long term expectations of being in state i and sub-state (i,j) are, respectively,

$$m_i = \lim_{k \to \infty} \Pr \{M_k = i\}, \text{ and}$$

$$m_{ij} = \lim_{t \to \infty} \Pr \{(x_t, y_t) = (i,j)\}.$$

References (b) and (c) show that

$$m_{ij} = b_{ij} m_i \Big/ \sum_{n=1}^{N} e_n m_n,$$

provided $b_{ij} > 0$, M is aperiodic and non-null, and $\sum_{n=1}^{\infty} e_n m_n$ converges. N is the total number of states. To say that M is aperiodic means every state in M can be returned to in any number of steps greater than one. To say that is non-null means that each $m_i > 0$.

In order for the steady state probabilities, m_i to be unique, M must be an irreducible Markov chain, i.e., all pairs of states of the chain must communicate. Thus, it must be possible to arrive at any state r from any other state s in a finite number of steps.

It can be shown that one characteristic of the steady state probabilities m_i is that

$$(m_1, m_2, \ldots, m_N) \ P = (m_1, m_2, \ldots, m_N)$$

where P is the matrix of transition probabilities mentioned above. This fact is used to determine the m_i's in the application that follows.

The interpretation of m_{ij} is discussed below.

APPLICATION

The states of a Markov chain M describing the events in an ASW engagement might be defined as follows:

1: holding no contact,
2: holding a valid contact, but not prosecuting it,
3: holding a false contact, but not prosecuting it,
4: prosecuting a valid contact, and
5: prosecuting a false contact.

It is assumed that the ASW engagement goes on through these five states indefinitely. (This is equivalent to assuming that the chain is non-null.) This situation may be approximated when one ASW aircraft relieves another on a barrier station. It is submitted that the computation of m_{5j} is of interest. This number represents the probability density of being in state (5,j) in the substate chain. For this application, it is the long term expectation of prosecuting a false contact for a length of time j. It is necessary that transition probabilities of P be estimated so that each m_i can be computed. These are then computed from the following sets of equations:

$$m_j = \sum_{i=1}^{N} m_i \, p_{ij} \text{, and}$$

$$\sum_{i=1}^{N} m_i = 1.$$

Further it is necessary to estimate the distributions of w_{ij}. In order for the (constant) set of transition probabilities p_{ij} and the waiting time distribution w_{ij} to have operational meaning, a constant submarine density or constant arrival rate must be assumed. It is not necessary to compute this density or arrival rate, however.

AN EXAMPLE

Suppose waiting times in the five states are retrieved and displayed as in table I, which shows, for example, that there were 3,096 occasions when no-contact was held for 1 to 10 minutes. Further, suppose that the frequencies of transition from state to state were as shown in figure 1. For example, state 1, holding no contact, was followed 9,288 times by state 2, holding a valid contact, and 3,096 times by state 3, holding a false contact. Hence the following transition probabilities are computed:

TABLE I

HYPOTHETICAL OBSERVED WAITING TIMES

Waiting time (minutes)	Holding no contact (state 1)	Holding a valid contact (state 2)	Holding a false contact (state 3)	Prosecuting a valid contact (state 4)	Prosecuting a false contact (state 5)
0	991	743	0	0	0
1-10	3096	2322	1238	1858	1032
11-20	3096	2322	929	1393	688
21-30	1858	1393	619	929	344
31-40	1857	1393	310	464	0
41-50	681	511	0	0	0
51-60	681	510	0	0	0
61-70	62	47	0	0	0
71-80	62	47	0	0	0
Totals	12384	9288	3096	4644	2064

Entries are the number of occurrences.

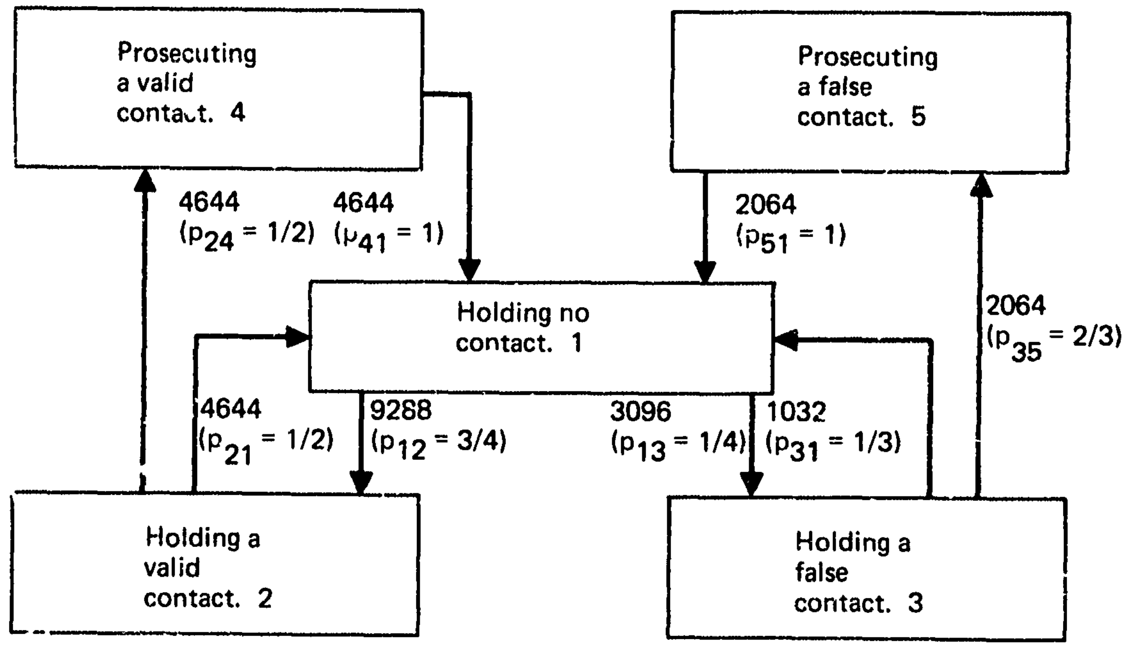

FIG. 1: SCHEMATIC OF TRANSITIONS

$$p_{12} = \frac{9{,}288}{12{,}384} = \frac{3}{4}, \qquad p_{31} = \frac{1{,}032}{3{,}096} = \frac{1}{3},$$

$$p_{13} = \frac{3{,}096}{12{,}384} = \frac{1}{4}, \qquad p_{35} = \frac{2{,}064}{3{,}096} = \frac{2}{3},$$

$$p_{21} = \frac{4{,}644}{9{,}288} = \frac{1}{2}, \qquad p_{41} = \frac{4{,}644}{4{,}644} = 1, \text{ and}$$

$$p_{24} = \frac{4{,}644}{9{,}288} = \frac{1}{2}, \qquad p_{51} = \frac{2{,}064}{2{,}064} = 1.$$

The various transition probabilities are displayed in matrix form in figure 2.

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} \end{pmatrix} = \begin{pmatrix} 0 & 3/4 & 1/4 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FIG. 2: THE TRANSITION MATRIX

The computation of $m_{5j} = b_{5j}\, m_5 \Big/ \sum_{n=1}^{5} e_n m_n$ goes as follows. First, the m_n's are calculated from:

$$m_1 + m_2 + m_3 + m_4 + m_5 = 1\ ,$$

$$m_1 = 0 \cdot m_1 + \frac{1}{2} \cdot m_2 + \frac{1}{3} \cdot m_3 + 1 \cdot m_4 + 1 \cdot m_5\ ,$$

$$m_2 = \frac{3}{4} \cdot m_1 + 0 \cdot m_2 + 0 \cdot m_3 + 0 \cdot m_4 + 0 \cdot m_5\ ,$$

$$m_3 = \frac{1}{4} \cdot m_1 + 0 \cdot m_2 + 0 \cdot m_3 + 0 \cdot m_4 + 0 \cdot m_5\ ,$$

$$m_4 + 0 \cdot m_1 + \frac{1}{2} \cdot m_2 + 0 \cdot m_3 + 0 \cdot m_4 + 0 \cdot m_5\ , \text{and}$$

$$m_5 = 0 \cdot m_1 + 0 \cdot m_2 + \frac{2}{3} \cdot m_3 + 0 \cdot m_4 + 0 \cdot m_5\ .$$

The result is:

$$m_1 = \frac{24}{61} = 0.393,$$

$$m_2 = \frac{18}{61} = 0.295,$$

$$m_3 = \frac{6}{61} = 0.098,$$

$$m_4 = \frac{9}{61} = 0.148, \text{ and}$$

$$m_5 = \frac{4}{61} = 0.066\ .$$

If the occurrences are assumed to be equally distributed among the minutes in each ten-minute interval of table I, the e_n's may be computed as follows:

$$e_1 = (0)(991/12384) + (1)(309.6/12384) + (2)(309.6/12384) + \ldots + (21)(185.8/12384) + \ldots + 80\,(6.2/12384).$$

The values $e_2, e_3, e_4,$ and e_5 are computed in a similar manner. The result of these computations is:

$$e_1 = 20.66$$

$$e_2 = 20.66$$

$$e_3 = 15.50$$

$$e_4 = 15.50\ , \text{and}$$

$$e_5 = 12.17\ ,$$

so that $m_5 \Big/ \sum_{n=1}^{5} e_n m_n = 0.066/18.83 = 0.0035$. Now it is necessary only to compute $0.0035 b_{5j}$ for each waiting time j from 1 to 30 (since $b_{5j} = 0$ for j 30). For example:

$$b_{5\,28} = (34.4/2064) + (34.4/2064). \text{ and}$$

$$m_{5\,28} = 0.0035\, b_{5\,28} = 0.000175\,.$$

The probability curve for m_{5j} is plotted in figure 3.

Suppose, that under the conditions to which the above computations apply, it is estimated that the ASW system is seriously degraded by the prosecution of a false contact only if the prosecution time exceeds 10 minutes. The probability that a false contact is being prosecuted and will continue to be prosecuted for 10 more minutes would be computed by summing m_{5j} over j equal to or greater than 10. In this case

$$\sum_{j=10}^{30} m_{5j} = 0.0174\,.$$

Note that this is not to be confused with the probability that a false contact is being prosecuted and that the total prosecution time will be 10 minutes or more. This latter probability in general will be larger, and in this case is about 0.03.

It should be pointed out that in practice one seldom has available as large a set of data in one experiment as shown in the hypothetical example of table I. Typically the waiting time distribution in one state is estimated from one experiment and the waiting time distribution for another state from another experiment. If this were not the case, the required probabilities could be computed by taking the appropriate sums and ratios directly from table I.

FADAP DATA

The transition probabilities can be estimated directly from FADAP data by simply observing the relative frequencies of the various transitions as was done in the example. One is restricted only in defining states that correspond to events as defined by reference (d). Examples of events include initial detection, no detection, false contact, regain contact, additional target detection, confirmation, localization, attack, change in contact classification, and target exposure change.

The data bank is structured so that many waiting time distributions, w_{ij}'s, can be estimated. There are provisions for obtaining the times that many events are initiated and terminated.

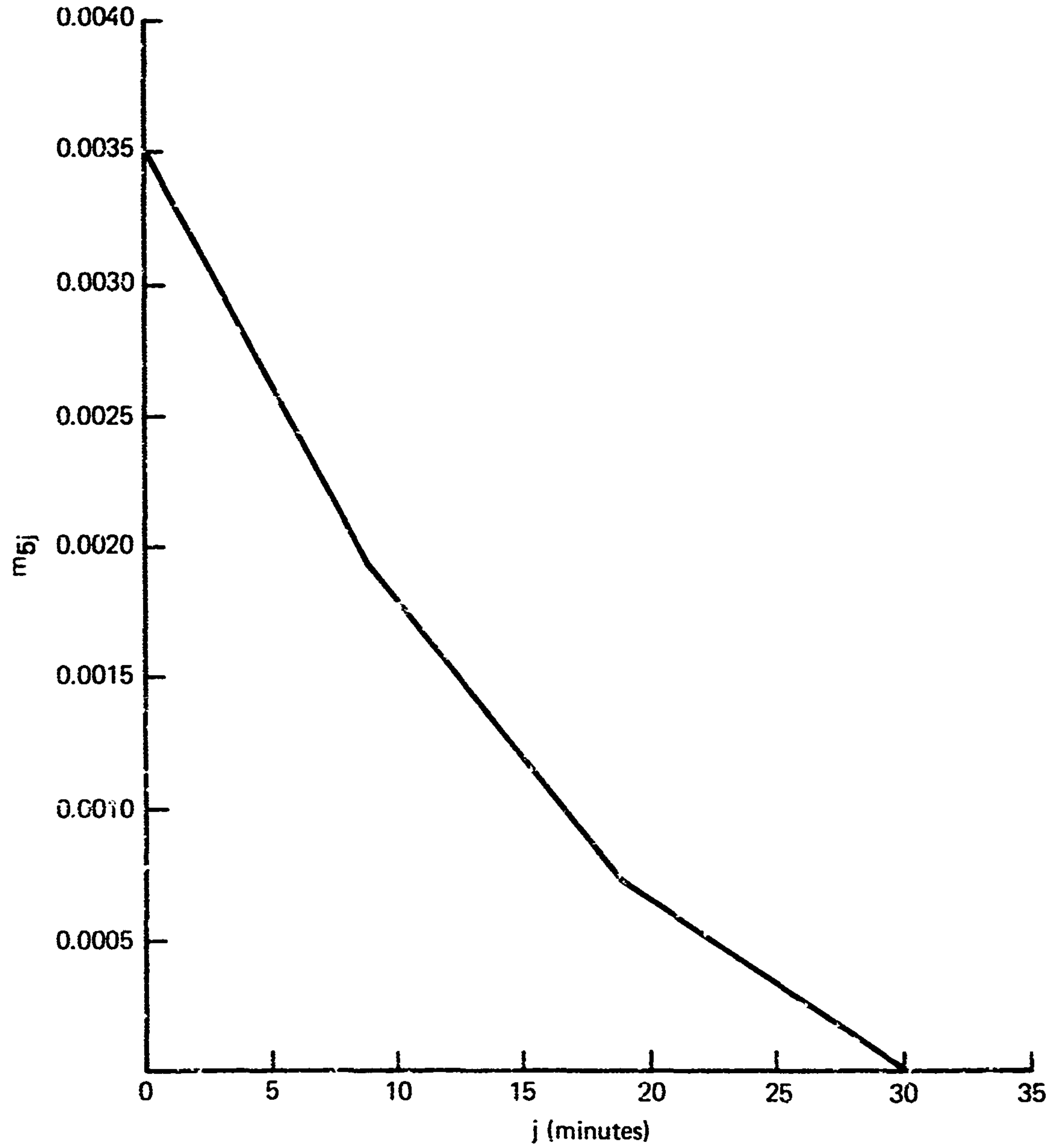

FIG. 3: PROBABILITY DENSITY FOR SUBSTATE (5, j)

LIMITATIONS

One limitation to the substate chain model is the range of choices for the states of the Markov chain M. The model does not allow for absorbing states, which are often very natural for describing ASW attack sequences. An absorbing state is a state that cannot be left once it has been entered $p_{ir} = 0$ when $i \neq r$). However, a way of handling waiting times when absorbing states are necessary is described in reference (e).

Most of the usual limitations of data collection apply. One major nuisance is missing data. Another is that the very process of carefully defining states excludes so many data that sample sizes chronically are small. To a degree this is being overcome by the continual addition of data from fleet exercises to the data bank. However, since different exercises test different hypotheses, some exercises may not include all the states defined by a chain. Also, equipment changes over the years can affect both transition probabilities and waiting times.

CONCLUSIONS

Despite the limitations cited above, it is felt that the FADAP data bank provides data structured in such a way that meaningful analysis can be achieved by the use of Markov chains. Further, in some cases Markov chains can be extended in such a way to account for waiting times in various states. In particular, it is felt that the probability that a false contact will be prosecuted for a given time conditional only on the structure of the whole ASW sequence of events can be computed.

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(d) Internal Navy Directive, 1965

(e) Memorandum for Director, OEG, (OEG) 548-68, "The Reduction of a Non-Markov Process to a Markov Process by the Method of Stages," 3 Oct 1968

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